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On Some Extremal Properties of Algebraic Polynomials

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Let p_n be an algebraic polynomial of degree n with real coefficients. Employing an inequality of G. Szegő, we study the extremal property of the integral $\int_{-1}^{1} (p_n'(x))^2 (1-x^2)\,dx$ subject to the constraint $\|p_n\|_{L_\infty} \leqslant 1$.

1. INTRODUCTION

In 1979 G. K. Kristiansen [3] proved the following extension of the Markov inequality. We may describe his main theorem as follows.

THEOREM A. *Let $n \geqslant 1$. Let $\phi: [0, n^2] \to \mathbb{R}$ be such that the function $(\phi(x)-\phi(0))/x$ is positive and nondecreasing. Let p_n be any real algebraic polynomial of degree n, such that $|p_n(x)| \leqslant 1$ for $-1 \leqslant x \leqslant 1$. Then the functional $\int_{-1}^{1} \phi(|p_n'(x)|)\,dx$ assumes its maximal value if and only if $p_n = T_n$ ($T_n(x) = \cos n\theta$, $\cos\theta = x$).*

In particular, it follows that if $q \in [1, \infty)$, then for every algebraic polynomial p_n of degree n, we have

$$\left(\int_{-1}^{1} |p_n'(x)|^q\,dx\right)^{1/q} \leqslant \left(\int_{-1}^{1} |T_n'(x)|^q\,dx\right)^{1/q} \max_{-1 \leqslant x \leqslant 1} |p_n(x)|. \quad (1.1)$$

Moreover, the equality is attained if and only if $p_n = \pm T_n$.

Remark. It may be noted that B. D. Bojanov [1] also obtained an elegant solution of (1.1).

Let H_n be the class of all real polynomials of degree n bounded by 1 on the interval $-1 \leqslant x \leqslant 1$. S. N. Bernstein (cf. [7, Theorem 1.22.3, p. 5]) proved that the supremum norm of $(1-x^2)^{1/2} p_n'(x)$ on $[-1, 1]$ for arbitrary $p_n \in H_n$ is maximum if $p_n = T_n$. In view of Theorem A, it may be

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asked if the L^q norm of $(1-x^2)^{1/2}\,p_n'(x)$ $(p_n \in H_n)$ is also maximized by T_n for all $q \in [1, \infty)$. The following theorem answers the above problem in the important case $q=2$. More precisely, we prove.

THEOREM 1. *If $p_n \in H_n$ then we have*

$$\int_{-1}^{1} (1-x^2)(p_n'(x))^2\,dx \leqslant n^2\left(1+\frac{1}{4n^2-1}\right) = \int_{-1}^{1} (1-x^2)(T_n'(x))^2\,dx \quad (1.2)$$

with equality only for $p_n = T_n$.

Our next theorem concerns the following extension of Theorem A.

THEOREM 2. *If $p_n \in H_n$, then we have*

$$\int_{-1}^{1} (p_n''(x))^2\,(1-x^2)^{-1/2}\,dx \leqslant \int_{-1}^{1} (T_n''(x))^2\,(1-x^2)^{-1/2}\,dx. \quad (1.3)$$

Remark. It is interesting to note that the corresponding extension of Theorem A to

$$\int_{-1}^{1} (p_n''(x))^2\,dx \leqslant \int_{-1}^{1} (T_n''(x))^2\,dx$$

under the assumption that $p_n \in H_n$ remains an unsolved problem.

2. PROOF OF THEOREM 1

Set

$$t_n(\theta) = p_n(\cos\theta) = p_n(x). \quad (2.1)$$

Then t_n is a purely cosine polynomial of degree $\leqslant n$. Further, we have

$$t_n'(\theta) = -(1-x^2)^{1/2}\,p_n'(x), \qquad t_n''(\theta) = (1-x^2)p_n''(x) - xp_n'(x). \quad (2.2)$$

From (2.1), (2.2), and integration by parts, we obtain

$$2\int_{-1}^{1} (1-x^2)(p_n'(x))^2\,dx$$

$$= \int_0^{\pi} ((t_n'(\theta))^2 - t_n(\theta)\,t_n''(\theta))\sin\theta\,d\theta - \int_0^{\pi} \cos\theta\, t_n(\theta)\,t_n'(\theta)\,d\theta.$$

A further integration by parts yields

$$\int_0^\pi \cos\theta t_n(\theta)\, t_n'(\theta)\, d\theta = -\left(\frac{t_n^2(0)+t_n^2(\pi)}{2}\right)+\frac{1}{2}\int_0^\pi t_n^2(\theta)\sin\theta\, d\theta.$$

We therefore obtain

$$\begin{aligned} 2\int_{-1}^{1} &(1-x^2)(p_n'(x))^2\, dx \\ &= \int_0^\pi ((t_n'(\theta))^2 - t_n(\theta)\, t_n''(\theta))\sin\theta\, d\theta \\ &\quad + \tfrac{1}{2}(t_n^2(\pi)+t_n^2(0)) - \tfrac{1}{2}\int_0^\pi t_n^2(\theta)\sin\theta\, d\theta. \end{aligned} \tag{2.3}$$

Next, we observe that

$$\begin{aligned} I_n &\equiv \int_0^\pi ((t_n'(\theta))^2 - t_n(\theta)\, t_n''(\theta))\sin\theta\, d\theta - \frac{1}{2}\int_0^\pi t_n^2(\theta)\sin\theta\, d\theta \\ &= \left(\frac{1}{2}-\frac{1}{2n^2}\right)\int_0^\pi [(t_n'(\theta))^2 + n^2 t_n^2(\theta)]\sin\theta\, d\theta \\ &\quad + \frac{1}{2}\int_0^\pi \left[\left(\frac{t_n'(\theta)}{n}\right)^2 n^2 + \left(\frac{t_n''(\theta)}{n}\right)^2\right]\sin\theta\, d\theta \\ &\quad + \frac{1}{2n^2}\int_0^\pi (t_n'(\theta))^2 \sin\theta\, d\theta - \frac{1}{2n^2}\int_0^\pi (n^2 t_n(\theta)+t_n''(\theta))^2\sin\theta\, d\theta. \end{aligned} \tag{2.4}$$

Let t_n be any real trigonometric polynomial of degree at most n such that $|t_n(\theta)| \leqslant 1$, $0 \leqslant \theta \leqslant 2\pi$, then we have

$$n^2 t_n^2(\theta) + t_n'^2(\theta) \leqslant n^2. \tag{2.5}$$

This inequality is due to G. Szegő [6, p. 64].

Applying Bernstein's inequality, we then obtain

$$\left(\frac{t_n'(\theta)}{n}\right)^2 n^2 + \left(\frac{t_n''(\theta)}{n}\right)^2 \leqslant n^2. \tag{2.6}$$

From (2.4), (2.5), (2.6), and

$$\int_0^\pi (t_n'(\theta))^2 \sin\theta\, d\theta = \int_{-1}^{1} (1-x^2)(p_n'(x))^2\, dx, \tag{2.7}$$

we have

$$I_n \leqslant \left(\frac{1}{2}-\frac{1}{2n^2}\right) 2n^2 + n^2 + \frac{1}{2n^2}\int_{-1}^{1} (1-x^2)(p_n'(x))^2\, dx$$
$$-\frac{1}{2n^2}\int_0^{\pi} (n^2 t_n(\theta)+t_n''(\theta))^2 \sin\theta\, d\theta. \tag{2.8}$$

Now, from (2.3), (2.4), and (2.8), we have

$$2\int_{-1}^{1} (1-x^2)(p_n'(x))^2\, dx$$
$$\leqslant \frac{1}{2}(t_n^2(0)+t_n^2(\pi)) + n^2\left(1-\frac{1}{n^2}\right)$$
$$+ n^2 + \frac{1}{2n^2}\int_{-1}^{1} (1-x^2)(p_n'(x))^2\, dx$$
$$-\frac{1}{2n^2}\int_0^{\pi} (n^2 t_n(\theta)+t_n''(\theta))^2 \sin\theta\, d\theta. \tag{2.9}$$

But (2.9) is equivalent to

$$\left(2-\frac{1}{2n^2}\right)\int_{-1}^{1} (1-x^2)(p_n'(x))^2\, dx \leqslant 2n^2 - \frac{1}{2}(1-t_n^2(0)) - \frac{1}{2}(1-t_n^2(\pi))$$
$$\leqslant 2n^2$$

from which (1.2) follows.

Let p_n be any polynomial of degree n belonging to H_n but different from the Chebyshev polynomial T_n. In this case equality is not possible in (2.9). To see this it is enough to note that the last term in the right hand side of (2.8) will be zero if and only if $t_n''(\theta)+n^2 t_n(\theta)=0$.

3. Proof of Theorem 2.

Before proving Theorem 2, we state and prove two auxillary lemmas. Let us denote by

$$x_i = \cos\frac{(2i-1)}{2n},\ i=1, 2, ..., n, \qquad y_i = \cos\frac{i\pi}{n},\ i=1, 2, ..., n-1, \tag{3.1}$$

the zeros of $T_n(x)=\cos n\theta$, $\cos\ \theta = x$, and $U_{n-1}(x)=\sin n\theta/\sin\theta$, $x=\cos\theta$, respectively.

LEMMA 3.1. *Let q_{n-1} be any algebraic polynomial of degree $\leqslant n-1$ such that*

$$|q_{n-1}(x_i)| \leqslant (1-x_i^2)^{-1/2}, \quad i=1, 2, ..., n, \tag{3.2}$$

where the x_i's are given by (3.1). *Then we have*

$$|q'_{n-1}(y_i)| \leqslant |U'_{n-1}(y_i)|, \qquad i=1, 2, ..., n-1 \tag{3.3}$$

and

$$|q'_{n-1}(1)| \leqslant |U'_{n-1}(1)|, \qquad |q'_{n-1}(-1)| \leqslant |U'_{n-1}(-1)|, \tag{3.4}$$

where the y_i's are as given in (3.1).

Proof. The proof of this lemma is analogous to [2, Lemma 1, p. 518]. By the Lagrange interpolation formula based on the zeros of T_n, we can represent any algebraic polynomial q_{n-1} of degree at most $n-1$ by

$$q_{n-1}(x) = \frac{1}{n} \sum_{i=1}^{n} q_{n-1}(x_i) \frac{T_n(x)}{x-x_i} (-1)^{i+1} (1-x_i^2)^{1/2}. \tag{3.5}$$

On differentiating both sides with respect to x and using (3.2), we have

$$|q'_{n-1}(y_j)| \leqslant \frac{1}{n} \sum_{i=1}^{n} \frac{|T_n(y_j)|}{(y_j-x_i)^2}. \tag{3.6}$$

Also, it is easy to see that

$$U'_{n-1}(y_j) = \frac{T_n(y_j)}{n} \sum_{i=1}^{n} \frac{1}{(y_i-x_i)^2}. \tag{3.7}$$

Therefore, we may conclude that

$$|q'_{n-1}(y_j)| \leqslant |U'_{n-1}(y_j)|, \qquad j=1, 2, ..., n-1.$$

This proves (3.3). The proof of (3.4) follows similarly and so we omit the details.

LEMMA 3.2. *Let q_{n-1} be an arbitrary algebraic polynomial of degree at most $n-1$ satisfying* (3.2). *Then*

$$\int_{-1}^{1} (q'_{n-1}(x))^2 (1-x^2)^{1/2}\, dx \leqslant \int_{-1}^{1} (U'_{n-1}(x))^2 (1-x^2)^{1/2}\, dx. \tag{3.8}$$

Proof. The proof of this lemma follows from Lemma 3.1 and a quadrature formula of Micchelli and Rivlin [4, formula (19), p. 12]. According to this quadrature formula for every polynomial p_{2n-1} of degree at most $2n-1$, we have

$$\int_{-1}^{1} p_{2n-1}(x)(1-x^2)^{-1/2}\,dx = \frac{\pi}{2n}\left[p_{2n-1}(1)+p_{2n-1}(-1)+2\sum_{i=1}^{n-1} p_{2n-1}(y_i)\right], \tag{3.9}$$

where the y_i's are as given in (3.1). From (3.3), (3.4), and (3.9), we have

$$\begin{aligned}\int_{-1}^{1} (q'_{n-1}(x))^2\,(1-x^2)^{-1/2}\,dx &\leqslant \frac{\pi}{2n}\left[(U'_{n-1}(1))^2+(U'_{n-1}(-1))^2+2\sum_{i=1}^{n-1}(U'_{n-1}(y_i))^2\right]\\ &=\int_{-1}^{1}(U'_{n-1}(x))^2\,(1-x^2)^{-1/2}\,dx.\end{aligned}$$

This proves Lemma 3.2. We are now in a position to prove Theorem 2.

Proof. Let $p_n \in H_n$. Then by Bernstein's inequality, it follows that

$$|p'_n(x)| \leqslant n(1-x^2)^{-1/2}, \qquad -1<x<1.$$

We define

$$q_{n-1}(x)=\frac{p'_n(x)}{n}.$$

Then clearly

$$|q_{n-1}(x)| \leqslant (1-x^2)^{-1/2}, \qquad -1<x<1.$$

Therefore the conditions in Lemma 3.2 are satisfied. Hence, we obtain from the lemma

$$\int_{-1}^{1} (q'_{n-1}(x))^2\,(1-x^2)^{-1/2}\,dx \leqslant \int_{-1}^{1} (U'_{n-1}(x))^2\,(1-x^2)^{-1/2}\,dx.$$

We can rewrite the above as

$$\begin{aligned}\int_{-1}^{1} (p''_n(x))^2\,(1-x^2)^{-1/2}\,dx &\leqslant n^2\int_{-1}^{1} (U'_{n-1}(x))^2\,(1-x^2)^{-1/2}\,dx\\ &=\int_{-1}^{1}(T''_n(x))^2\,(1-x^2)^{-1/2}\,dx.\end{aligned}$$

This proves Theorem 2.

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